On the Modified Leverrier-Faddeev Algorithm

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ABSTRACT

Properties of an eigenmatrix proposed by Faddeev and an extension thereof investigated by Gower are deduced from consideration of the adjoint and reduced adjoint of the characteristic matrix $sI - A$.

1. INTRODUCTION

In a recent paper in this journal [1], Gower investigated properties of an eigenmatrix proposed by Faddeev and obtained an extension to the repeated eigenvalue case. His methods rested essentially on use of the Jordan canonical form. We give here a separate analysis in which the principal tool is matrices with polynomial entries (λ-matrices). It is shown that Faddeev's eigenmatrix is related to the adjoint of the characteristic matrix $sI - A$, while Gower's extension is related to the reduced adjoint of that matrix. From this point of view it is not necessary to treat separately the zero eigenvalue (when present).

A decomposition of the reduced adjoint with respect to a generalized eigenspace is obtained.

2. THE ADJOINT

Given the monic nth degree polynomial

$$s^n + p_1 s^{n-1} + \cdots + p_n,$$

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define a sequence of polynomials $c_i(s)$ recursively by

$$c_0(s)=1,$$
$$c_j(s)=sc_{j-1}(s)+p_j, \qquad 1\leqslant j\leqslant n, \tag{1}$$

and note that $c_n(s)$ is the given polynomial. It is readily verified from the identity

$$(sI-A)\,\mathrm{adj}(sI-A)=c_n(s)I$$

that

$$\mathrm{adj}(sI-A)=\sum_{i=0}^{n-1} c_i(s)A^{n-i-1}$$

when A is an $n\times n$ matrix whose characteristic polynomial is $c_n(s)$. Set

$$Z(s)=A\,\mathrm{adj}(sI-A)=s\,\mathrm{adj}(sI-A)-c_n(s)I$$
$$=\sum_{i=0}^{n-1} c_i(s)A^{n-i}. \tag{2}$$

Now consider

$$Y(s)=\sum_{i=0}^{n-1} s^{n-i-1}Y_i$$

with the $n\times n$ matrices Y_i given by the modified Leverrier sequence

$$Y_0=A,$$
$$Y_j=AY_{j-1}+p_jA, \qquad 1\leqslant j\leqslant n,$$

where $p_j=-j^{-1}\operatorname{tr}Y_{j-1}$. Choose s to be any element λ of the spectrum of A. Then, as cited in [1], D. K. Faddeev proposed that under the circumstance of distinct eigenvalues $Y(\lambda)$ is a nonnull matrix all of whose columns are in the eigenspace associated with λ. We show these properties of $Y(\lambda)$, when $\lambda\neq 0$, to be an immediate consequence of

THEOREM 1. $Y(s) = Z(s)$.

Proof. It first should be noted that the Y_i can be written in terms of the polynomial set $c_j(s)$ as

$$\begin{aligned}
Y_0 &= A = Ac_0(A),\\
Y_1 &= A(Y_0 + p_1 I) = A[Ac_0(A) + p_1 I] = Ac_1(A),\\
&\vdots\\
Y_{n-1} &= Ac_{n-1}(A),\\
Y_n &= Ac_n(A) = 0.
\end{aligned}$$

Then, after appealing to the relations (1) and rearranging terms, we obtain

$$Y(s) = \sum_{i=0}^{n-1} s^{n-i-1} Ac_i(A) = \sum_{i=0}^{n-1} s^{n-i} c_i(A) - c_n(s) I. \tag{3}$$

Form the two variable polynomial function

$$f(s,t) = \sum_{i=0}^{n} s^{n-i} c_i(t),$$

and note that this function is symmetric in s and t. One way to show this is to compute the partial derivatives at (0,0). Let $D^{(\mu,\nu)}$ denote μ derivatives on s and ν on t. Straightforward computation shows $D^{(\mu,\nu)} f(0,0) = \mu!\ c_{n-\mu}^{(\nu)}(0)$. Since

$$c_{n-\mu}(t) = t^{\nu} c_{n-\mu-\nu}(t) + (\nu - 1)\text{th degree polynomial},$$

$c_{n-\mu}^{(\nu)}(0) = \nu!\ c_{n-\mu-\nu}(0)$ and symmetry follows. (Indeed, it can be shown that $f(s,t) = [sc_n(s) - tc_n(t)]/(s-t)$.) Since A and sI commute, the function f is uniquely defined on the pair of arguments sI, A and $f(sI, A) = f(A, sI)$, i.e.,

$$\sum_{i=0}^{n} s^{n-i} c_i(A) = \sum_{i=0}^{n} c_i(s) A^{n-i}.$$

From Equation (3), the left hand sum is $Y(s)+c_n(s)I$. The desired result is obtained by noting that the Cayley-Hamilton theorem and (1) reduce the right hand side to $s\,\mathrm{adj}(sI-A)$. ■

Compute

$$AY(s)=(A-sI+sI)Z(s)=sZ(s)-A(sI-A)\,\mathrm{adj}(sI-A)$$
$$=sY(s)-c_n(s)A.$$

Clearly this implies $AY(\lambda)=\lambda Y(\lambda)$ for any λ in the spectrum of A.

If B is an $n\times n$ matrix, then rank(adj B) is $n,1,0$ according as rank B is n, $n-1$, or less than $n-1$. Consequently, for $\lambda\neq 0$, $Y(\lambda)=Z(\lambda)$ is nonnull, since λ is unrepeated. However, we should note that it is the *geometric* multiplicity of λ rather than its *algebraic* multiplicity that is relevant in drawing this conclusion. Calculation of $Z(s)$ for a 2×2 companion matrix gives concrete evidence that $Y(\lambda)$ can be nonnull even in the presence of a repeated eigenvalue.

We note that $Y(0)=0$ because of the multiplier s of $\mathrm{adj}(sI-A)$ in $Z(s)$ [equivalently, $Y(0)=Y_{n-1}=Ac_{n-1}(A)=c_n(A)-p_nI=0$], and no information about eigenvectors associated with the zero eigenvalue are obtainable from the Faddeev matrix. Since $\mathrm{adj}(sI-A)$ does give information in all cases (of geometric multiplicity one) and can also be developed by a modified Leverrier algorithm, it may be suggested that the eigenmatrix of Faddeev is ill proposed.

3. THE REDUCED ADJOINT

Let $d(s)$ be the gcd of the n^2 entries of $\mathrm{adj}(sI-A)$. Then, letting $m(s)$ denote the minimum polynomial of A and $\mathrm{Radj}(sI-A)$ the reduced adjoint of A, we have [2]

$$c_n(s)=d(s)m(s)$$

and

$$\mathrm{adj}(sI-A)=d(s)\,\mathrm{Radj}(sI-A).$$

By Theorem 1,

$$Y(s) = s\,\mathrm{adj}(sI - A) - c_n(s)I.$$

It follows that $d(s)$ divides $Y(s)$ and so

$$X(s) = s\,\mathrm{Radj}(sI - A) - m(s)I, \tag{4}$$

where $Y(s) = d(s)X(s)$. Since

$$(sI - A)\,\mathrm{Radj}(sI - A) = m(s)I - m(A) = m(s)I, \tag{5}$$

left multiplication of (4) by $sI - A$ gives

$$AX(s) = sX(s) - m(s)A. \tag{6}$$

Let $m(s) = s^m + q_1 s^{m-1} + \dots + q_m$. By (4) and (5), $X(s)$ is of degree $m-1$ and so it will have the expansion

$$X(s) = \sum_{i=0}^{m-1} s^{m-i-1} X_i'.$$

By (6), the X_i' satisfy

$$\begin{aligned} X_0' &= A, \\ X_j' &= AX_{j-1}' + q_j A, \qquad 1 \leqslant j \leqslant m-1, \\ 0 &= AX_{m-1}' + q_m A, \end{aligned}$$

and consequently $X(s)$ corresponds to the matrix introduced by Gower.

For λ in the spectrum of A, (6) tells us that

$$AX(\lambda) = \lambda X(\lambda).$$

For $\lambda \neq 0$, (4) tells us that $X(\lambda)$ is nonnull—otherwise the reduced adjoint would not be reduced. Set $m(s) = (s-\lambda)^k r(s)$ with $r(\lambda) \neq 0$. Let A_1, A_2 denote the restrictions of A to the invariant subspaces V_1, V_2 which are the nullspaces of $(A - \lambda I)^k$ and $r(A)$ respectively. The rank of $\mathrm{Radj}(\lambda I - A)$ will follow directly from the following theorems.

THEOREM 2.

$$\operatorname{Radj}(sI-A)=r(A)\operatorname{Radj}(sI-A_1)+(s-\lambda)^k\operatorname{Radj}(sI-A_2).$$

Proof. Denote the right hand side by $R(s)$. Then

$$\begin{aligned}(sI-A)R(s)&=r(A)[(sI-A)\operatorname{Radj}(sI-A_1)]\\&\quad+(s-\lambda)^k[(sI-A)\operatorname{Radj}(sI-A_2)]\\&=r(A)\left[(s-\lambda)^kI-(A-\lambda I)^k\right]+(s-\lambda)^k[r(s)I-r(A)]\\&=m(s)I,\end{aligned}$$

since $(s-\lambda)^k$ and $r(s)$ are the minimum polynomials on V_1, V_2 respectively. But the regular polynomial matrix $m(s)I$ has a unique right quotient under division by $sI-A$. Consequently $R(s)=\operatorname{Radj}(sI-A)$ by (5). ■

THEOREM 3.

$$\operatorname{Radj}(\lambda I-A_1)=(A-\lambda I)^{k-1}.$$

Proof. If B has minimum polynomial $m_k(s)=s^k+b_1s^{k-1}+\ldots+b_k$ and if a set of polynomials $m_j(s)$ is defined by $m_0(s)=1$ and $m_j(s)=sm_{j-1}(s)+b_j$ for $1\leqslant j\leqslant k$, then it follows as in the first part of this paper that

$$\operatorname{Radj}(sI-B)=\sum_{j=0}^{k-1}m_j(s)B^{k-j-1}=\sum_{j=0}^{k-1}s^{k-j-1}m_j(B). \tag{7}$$

When $m_k(s)=(s-\lambda)^k$, it is an easy induction proof to show that

$$m_j(\lambda)=(-1)^j\binom{k-1}{j}\lambda^j.$$

The result then follows. ■

From the above theorems we obtain

$$\operatorname{Radj}(\lambda I-A)=r(A)(A-\lambda I)^{k-1}. \tag{8}$$

It is now manifest that only the generalized eigenvectors of λ of highest grade survive annihilation under the action of $\mathrm{Radj}(\lambda I - A)$, and consequently that the rank of this matrix is equal to the number of maximal sized Jordan blocks associated with λ. We observe, as at the end of Section 1, that it is not necessary to treat the zero eigenvalue separately when using the reduced adjoint. Moreover, the second sum in (7) shows that it can be developed from the minimum polynomial as was the matrix X of Gower.

REFERENCES

1 J. C. Gower, A modified Leverrier-Faddeev algorithm for matrices with multiple eigenvalues, *Linear Algebra Appl.* 31:61–70 (1980).
2 P. Lancaster, *Theory of Matrices*, Academic, New York, 1969, pp. 125–138.

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